

On the Quantum Mechanics for One Photon

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Abstract

This paper revisits the quantum mechanics for one photon from the modern viewpoint and by the geometrical method. Especially, besides the ordinary (rectangular) momentum representation, we provide an explicit derivation for the other two important representations, called the cylindrically symmetrical representation and the spherically symmetrical representation, respectively. These other two representations are relevant to some current photon experiments in quantum optics. In addition, the latter is useful for us to extract the information on the quantized black holes. The framework and approach presented here are also applicable to other particles with arbitrary mass and spin, such as the particle with spin $\frac{1}{2}$.

1 Introduction

From the modern viewpoint, relativistic quantum mechanics originates from the natural marriage of special relativity and quantum theory: The Hilbert space for one particle quantum wave functions forms the unitary representation of the Poincare group, which is the isometric transformation group of the Minkowski spacetime. Especially, as realized on the Minkowski spacetime, the quantum wave

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functions need to satisfy the field equation of motion[1, 2, 3, 4, 5]. Not only does it provide a basis for relativistic quantum field, i.e., the quantum field operator is just defined on the Fock space associated with the Hilbert space of one particle states, but relativistic quantum mechanics itself is of significance in those cases which do not involve particle creation and annihilation, such as free propagations for in states before interaction and out states after interaction. It is here that relativistic quantum mechanics demonstrates its most striking properties such as quantum superposition and quantum entanglement, and thus acquires many invaluable applications such as quantum information and quantum computation[6]. Moreover, the information on the interaction can be extracted by comparing out states with in states.

Obviously, light occupies a special position in our attempts to understand nature both relativistically and quantum mechanically. It was light that initiated the great birth of both special relativity and quantum theory. Furthermore, the quantum mechanics on the photon and its interaction with matters has been developed into an individual discipline with wide applications, now called quantum optics[7]. By the geometric method, this paper is mainly intended to revisit the quantum mechanics for one photon from the modern viewpoint mentioned in the beginning. In particular, besides the momentum representation, we explicitly provide the other two important representations, i.e., the cylindrically symmetrical representation and the spherically symmetrical one. These other two representations are very relevant to some of current research in quantum optics[8, 9, 10, 11]. Especially, the latter is significant to the multi-pole radiation and electromagnetic scattering around such a central potential as the Schwarzschild black hole. Furthermore, based on the Hod's corresponding principle, it acquires a new application in the quantized black holes[12].

It is worth noting that although all the three representations are known¹, they are treated here in a uniform framework and on an equal footing. Especially, for the derivation of the later two representations, our method is obviously different from that used before[13, 14]. We here employ the spin weighted harmonics functions and the corresponding spin weighted raising(lowering) operators, which have advantage of providing a straightforward and unified formalism applicable to particles of any spin.

The paper is organized as follows. In the next section, we construct the Hilbert space for one photon states from the solutions to the Maxwell equation. Based on the Killing field realization of the Poincare Lie algebra, Section 3 well defines the relevant conserved observables on the Hilbert space for one photon states, which is thus indicated to form the unitary representation of the Poincare group. The three representations are presented in Section 4, where the explicit derivation is given. We conclude with some implications and extensions in Section 5.

Our notation and conventions follow those in [3]. In particular, the index is raised or lowered by

¹The first and third can be found in many of advanced textbooks such as [13], but the second was obtained for the first time in [14].

the Minkowski metric η_{ab} . We denote the covariant derivative and volume element compatible with the metric by ∇_a and ϵ_{abcd} respectively. The d'Alembertian is defined as $\square = \nabla_a \nabla^a$. The Lorentz coordinate system is specially denoted by $\{x^\mu | \mu = 0, 1, 2, 3\}$, and the spatial vectors are indicated by letters in boldface.

2 The Hilbert Space for One Photon States

Start with the source free Maxwell equation on the Minkowski spacetime

$$\begin{aligned}\nabla_{[a} F_{bc]} &= 0, \\ \nabla^a F_{ab} &= 0,\end{aligned}\tag{1}$$

where F_{ab} is a skew tensor field, called field strength. It is obvious that the solutions to the Maxwell equation form a complex vector space, denoted by H . (More precisely, we define H to be the complex vector space of solutions which vanish rapidly at spatial infinity.) To introduce an inner product on our complex vector space, we first define a conserved current as

$$j_a[A, A'] = i[\bar{F}_{ab} A'^b - \bar{A}^b F'_{ab}],\tag{2}$$

where A_a is the vector potential, satisfying

$$F_{ab} = 2\nabla_{[a} A_{b]}.\tag{3}$$

Whence the inner product can be defined as

$$(F, F') = (A, A') = \int_{\Sigma} j^a[A, A'] \epsilon_{abcd}.\tag{4}$$

Note that the conservation of $j_a[A, A']$ implies that this inner product is independent of choice of the Cauchy surface Σ^2 . Thus, for the later convenience, we choose the surface of constant x^0 as Σ once and for all. Moreover, Eqn.(4) can be written as

$$(F, F') = (A, A') = \int_{\Sigma} \left(\frac{\partial}{\partial x^0}\right)^a j_a[A, A'] \tilde{\epsilon}_{bcd},\tag{5}$$

where $\tilde{\epsilon}_{bcd} = (\frac{\partial}{\partial x^0})^a \epsilon_{abcd}$ is the induced spatial volume element on Σ .

²This point also implies the unitarity of the evolution of source free fields.

In addition, by Eqn.(2), Eqn.(5), and the second part of Eqn.(1), the Stokes theorem shows that the inner product is invariant under gauge transformations

$$\begin{aligned} A_a &\rightarrow A_a + \nabla_a \Lambda, \\ A'_a &\rightarrow A'_a + \nabla_a \Lambda', \end{aligned} \quad (6)$$

where Λ and Λ' are both arbitrary scalar fields. However, this inner product is not always non-negative on our whole complex vector space. We next restrict H to its sub-vector space which guarantees the non-negativity of the above inner product. We denote this sub-vector space by H^+ , which is just the Hilbert space for one photon states.

3 Conserved Observables from The Poincare Lie Algebra

As is well known, the Poincare Lie algebra can be realized by the Killing vector fields on the Minkowski spacetime as follow

$$\begin{aligned} P_\mu &= i\left(\frac{\partial}{\partial x^\mu}\right)^a, \\ M_{\mu\nu} &= i[x_\mu\left(\frac{\partial}{\partial x^\nu}\right)^a - x_\nu\left(\frac{\partial}{\partial x^\mu}\right)^a]. \end{aligned} \quad (7)$$

According to the fact that the covariant derivative commutes with the Lie derivatives via Killing vector fields, the operators from the Poincare Lie algebra, i.e.

$$\begin{aligned} \hat{P}^\mu F_{ab} &= \mathcal{L}_{P^\mu} F_{ab}, \\ \hat{M}_{\mu\nu} F_{ab} &= \mathcal{L}_{M_{\mu\nu}} F_{ab}, \end{aligned} \quad (8)$$

are well defined on H . Moreover, it can be also shown that they are well defined on the Hilbert space for one photon states indeed³. Later, employing the Leibnitz rule, the conservation of $j_a[A, A']$, and the Stokes theorem, we find that the above operators are hermitian with respect to the inner product (5). In addition, since the inner product (5) is independent of the choice of Σ , the above operators is also conserved observables. Furthermore, taking into account $[\mathcal{L}_u, \mathcal{L}_v] = \mathcal{L}_{[u, v]}$ with u and v arbitrary vector fields, we can obtain

$$[\hat{P}_\mu, \hat{P}_\nu] = 0, \quad (9)$$

$$[\hat{P}_\mu, \hat{M}_{\rho\sigma}] = 2i\eta_{\mu[\rho}\hat{P}_{\sigma]}, \quad (10)$$

³It seems easier to prove in the rectangular momentum representation.

$$[\hat{M}_{\mu\nu}, \hat{M}_{\rho\sigma}] = 2i(\eta_{\mu[\rho}\hat{M}_{\sigma]\nu} - \eta_{\nu[\rho}\hat{M}_{\sigma]\mu}). \quad (11)$$

Here, \hat{P}^μ is the four-momentum operator. By Eqn(1), we have

$$\hat{P}_\mu \hat{P}^\mu = -\square = 0, \quad (12)$$

which shows that the eigenvalue of the four-momentum operator is null. Furthermore, $\{\hat{L}_1 \equiv \hat{M}_{23}, \hat{L}_2 \equiv \hat{M}_{31}, \hat{L}_3 \equiv \hat{M}_{12}\}$ are the total angular momentum operators.

We next introduce the Pauli-Lubanski spin vector operator

$$\hat{S}_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\hat{P}^\nu\hat{M}^{\rho\sigma}. \quad (13)$$

Resorting to Eqn.(1) and after a straightforward calculation, we can obtain⁴

$$\hat{S}_\mu = \hat{P}_\mu \hat{S}, \quad (14)$$

where \hat{S} is the helicity operator, defined by

$$\hat{S}F_{ab} = (-i)^*F_{ab} = -\frac{i}{2}\epsilon_{abcd}F^{cd}. \quad (15)$$

Based on the fact that the Lie derivatives via Killing vector fields annihilate the volume element, \hat{S} commutes with both \hat{P}_μ and $\hat{M}_{\mu\nu}$. Furthermore, we have

$$\hat{S}^2 = 1, \quad (16)$$

which implies that the possible eigenvalue of the helicity operator takes ± 1 .

4 Three Representations in the Coulomb Gauge

In this section, we shall employ the vector potential in the Coulomb gauge. In terms of the vector potential, the Maxwell equation can be written as

$$\square A_a = 0, \quad (17)$$

where the Coulomb gauge

$$\begin{aligned} \nabla^a A_a &= 0, \\ \left(\frac{\partial}{\partial x^0}\right)^a A_a &= 0, \end{aligned} \quad (18)$$

⁴The reader is suggested to follow the steps described in [15].

has been employed. In this case, the inner product (5) is equivalent to

$$(F, F') = (A, A') = \int_{\Sigma} \left(\frac{\partial}{\partial x^0} \right)^a j'_a [A, A'] \tilde{\epsilon}_{bcd}, \quad (19)$$

where the conserved current

$$j'_a [A, A'] = i [\nabla_a \bar{A}_b] A'^b - \bar{A}^b \nabla_a A'_b. \quad (20)$$

Later, according to the commutation relations in the last section, we can choose $\{\hat{P}^1, \hat{P}^2, \hat{P}^3, \hat{S}\}$ as a complete observable set, which forms the ordinary rectangular momentum representation. Similarly, the complete observable set $\{\hat{P}^0, \hat{P}^3, \hat{L}_3, \hat{S}\}$ forms the cylindrically symmetrical representation, and $\{\hat{P}^0, \hat{\mathbf{L}}^2, \hat{L}_3, \hat{S}\}$ forms the spherically symmetrical representation.

4.1 The Rectangular Momentum Representation

Since the details of the rectangular momentum representation have appeared in the literature, we will only recall the main results without entering into the explicit derivations. Firstly, according to Eqn.(17) and Eqn.(18), any vector potential can be written as

$$\begin{aligned} A_a(x) &= \frac{1}{\sqrt{(2\pi)^3}} \left\{ \int_{p^0 > 0} \frac{d^3 \mathbf{p}}{p^0} [\dot{A}_+(\mathbf{p})(\varepsilon^+)_a(\mathbf{p}) + \dot{A}_-(\mathbf{p})(\varepsilon^-)_a(\mathbf{p})] e^{-ip_b x^b} \right. \\ &\quad \left. + \int_{p^0 < 0} \frac{d^3 \mathbf{p}}{p^0} [\dot{A}_+(\mathbf{p})(\varepsilon^+)_a(\mathbf{p}) + \dot{A}_-(\mathbf{p})(\varepsilon^-)_a(\mathbf{p})] e^{-ip_b x^b} \right\}. \end{aligned} \quad (21)$$

Here $p^a = p^u (\frac{\partial}{\partial x^\mu})^a$ is a constant real null vector field, and $x^a = x^\mu (\frac{\partial}{\partial x^\mu})^a$ is the position vector field. In addition, $(\varepsilon^\pm)_a(\mathbf{p})$ are constant null fields and complex conjugate with each other, satisfying

$$\begin{aligned} p^a (\varepsilon^\pm)_a(\mathbf{p}) &= 0, \\ \left(\frac{\partial}{\partial x^0} \right)^a (\varepsilon^\pm)_a(\mathbf{p}) &= 0, \\ \epsilon_{abcd} &= -\frac{i}{p^0} (dx^0)_a \wedge p_b \wedge (\varepsilon^+)_c(\mathbf{p}) \wedge (\varepsilon^-)_d(\mathbf{p}). \end{aligned} \quad (22)$$

Later, substituting Eqn.(21) into Eqn.(19), we have

$$\begin{aligned} (F, F') &= (A, A') = \int d^3 \mathbf{x} \left(\frac{\partial}{\partial x^0} \right)^a j'_a [A, A'] \\ &= 2 \left\{ \int_{p^0 > 0} \frac{d^3 \mathbf{p}}{p^0} [\bar{A}_+(\mathbf{p}) \dot{A}'_+(\mathbf{p}) + \bar{A}_-(\mathbf{p}) \dot{A}'_-(\mathbf{p})] \right. \\ &\quad \left. + \int_{p^0 < 0} \frac{d^3 \mathbf{p}}{p^0} [\bar{A}_+(\mathbf{p}) \dot{A}'_+(\mathbf{p}) + \bar{A}_-(\mathbf{p}) \dot{A}'_-(\mathbf{p})] \right\}. \end{aligned} \quad (23)$$

Whence H^+ is just the positive energy solutions to the Maxwell equation, as is also what we expect. Therefore, we shall restrict us to the case of $p^0 > 0$ in all of the following discussions. Furthermore, the orthonormal basis for H^+ in the rectangular momentum representation is given by

$$|\mathbf{p}, s = \pm 1\rangle = \frac{1}{\sqrt{(2\pi)^3}} \frac{1}{\sqrt{2p^0}} (\varepsilon^\pm)_a(\mathbf{p}) e^{-ip_b x^b}, \quad (24)$$

where \mathbf{p} is the eigenvalue of three-momentum operator, and s is the eigenvalue of the helicity operator.

4.2 The Cylindrically Symmetrical Representation

It is convenient to provide the cylindrically symmetrical representation in the cylindrical coordinate system, i.e.

$$\begin{aligned} x^0 &= t, \\ x^1 &= \varrho \cos \phi, \\ x^2 &= \varrho \sin \phi, \\ x^3 &= z. \end{aligned} \quad (25)$$

Whence the Minkowski metric reads

$$ds^2 = dt^2 - dz^2 - d\varrho^2 - \varrho^2 d\phi^2, \quad (26)$$

and

$$\begin{aligned} P^0 &= i\left(\frac{\partial}{\partial t}\right)^a, \\ P^3 &= -i\left(\frac{\partial}{\partial z}\right)^a, \\ L_3 &= -i\left(\frac{\partial}{\partial \phi}\right)^a. \end{aligned} \quad (27)$$

Define a pair of null covariant vector fields as

$$(\varepsilon^\mp)_a = \frac{1}{\sqrt{2}} [(d\varrho)_a \pm i\varrho(d\phi)_a], \quad (28)$$

then according to the second part in Eqn.(18), any vector potential can be written as

$$A_a = A_z(dz)_a + A_-(\varepsilon^-)_a + A_+(\varepsilon^+)_a, \quad (29)$$

where, A_z has spin weight 0, A_- with spin weight -1 , and A_+ with spin weight 1 [16]. From

$$\begin{aligned}\nabla_a(dz)_b &= 0, \\ \nabla_a(\varepsilon^-)_b &= \frac{1}{\sqrt{2}\varrho}[(\varepsilon^+)_a - (\varepsilon^-)_a](\varepsilon^-)_b, \\ \nabla_a(\varepsilon^+)_b &= \frac{1}{\sqrt{2}\varrho}[(\varepsilon^-)_a - (\varepsilon^+)_a](\varepsilon^+)_b,\end{aligned}\tag{30}$$

it can be shown that the Maxwell equation reads

$$\begin{aligned}\square A_a &= (-\bar{\partial}\partial A_z + \frac{\partial^2 A_z}{\partial t^2} - \frac{\partial^2 A_z}{\partial z^2})(dz)_a \\ &\quad + (-\bar{\partial}\partial A_- + \frac{\partial^2 A_-}{\partial t^2} - \frac{\partial^2 A_-}{\partial z^2})(\varepsilon^-)_a \\ &\quad + (-\bar{\partial}\partial A_+ + \frac{\partial^2 A_+}{\partial t^2} - \frac{\partial^2 A_+}{\partial z^2})(\varepsilon^+)_a = 0,\end{aligned}\tag{31}$$

together with

$$\nabla^a A_a = -\frac{\partial A_z}{\partial z} + \frac{1}{\sqrt{2}}(\bar{\partial}A_- + \bar{\partial}A_+) = 0.\tag{32}$$

Here ∂ , $\bar{\partial}$ are operators acting on a quantity f with spin weight n , i.e.

$$\begin{aligned}\partial f &= -(\frac{\partial}{\partial \varrho} + \frac{i}{\varrho} \frac{\partial}{\partial \phi} - \frac{n}{\varrho})f, \\ \bar{\partial} f &= -(\frac{\partial}{\partial \varrho} - \frac{i}{\varrho} \frac{\partial}{\partial \phi} + \frac{n}{\varrho})f.\end{aligned}\tag{33}$$

Then, ∂f and $\bar{\partial} f$ have spin weight $n+1$ and $n-1$, respectively[16]. Later, it is easy to check that the Lie derivatives of $\{(dz)_a, (\varepsilon^\mp)_a\}$ via $\{P^0, P^3, L_3\}$ all vanish, thus the simultaneous eigensolutions of $\{\hat{P}^0, \hat{P}^3, \hat{L}_3\}$ to Eqn.(31) with the corresponding eigenvalue $\{p^0, p^3, m\}$ must take the form

$$\begin{aligned}A_z &= a_{00}Z_{\alpha m}(\varrho, \phi)e^{-i(p_0 t + p_3 z)}, \\ A_- &= a_{-1}Z_{\alpha m}(\varrho, \phi)e^{-i(p_0 t + p_3 z)}, \\ A_+ &= a_{+1}Z_{\alpha m}(\varrho, \phi)e^{-i(p_0 t + p_3 z)},\end{aligned}\tag{34}$$

where a_0 , a_- , and a_+ are all constant coefficients; ${}_n Z_{\alpha m}$ is the spin-weighted cylindrical harmonics with spin weight n such that[16]

$$\begin{aligned}\partial_n Z_{\alpha m} &= \alpha_{n+1}Z_{\alpha m}, \\ \bar{\partial}_n Z_{\alpha m} &= -\alpha_{n-1}Z_{\alpha m}, \\ \hat{L}_{3n} Z_{\alpha m} &= m_n Z_{\alpha m}\end{aligned}\tag{35}$$

with $\alpha = \sqrt{p_0^2 - p_3^2}$. Moreover, by the boundary condition, here p_3 is a real constant, and

$${}_nZ_{\alpha m} = J_{m+n}(\alpha \varrho) e^{im\phi}, \quad (36)$$

where J_{m+n} is the first kind of Bessel function of order $m+n$ with $\alpha \geq 0$ and m an integer.

Substituting Eqn.(34) into Eqn.(32), we have

$$ip_3 a_0 + \frac{\alpha}{\sqrt{2}}(a_- - a_+) = 0. \quad (37)$$

Next combine it with the eigenequations of the helicity operator, i.e.

$$\begin{aligned} ip_0 a_0 - s \frac{\alpha}{\sqrt{2}}(a_- + a_+) &= 0, \\ i(p_0 - sp_3)a_+ + s \frac{\alpha}{\sqrt{2}}a_0 &= 0, \\ i(p_0 + sp_3)a_- + s \frac{\alpha}{\sqrt{2}}a_0 &= 0, \end{aligned} \quad (38)$$

where $s = \pm 1$ is the eigenvalue of the helicity operator. Thus we have

$$\begin{aligned} a_- &= \frac{isa_0}{\sqrt{2}\alpha}(p_0 - sp_3), \\ a_+ &= \frac{isa_0}{\sqrt{2}\alpha}(p_0 + sp_3). \end{aligned} \quad (39)$$

Furthermore, note

$$\int_0^\infty d\varrho \varrho J_m(\alpha \varrho) J_m(\alpha' \varrho) = \frac{1}{\alpha} \delta(\alpha - \alpha') \quad (40)$$

with $\alpha \leq \alpha'$ [20]. Then it follows that the orthonormal basis with respect to the inner product (19) in the cylindrically symmetrical representation reads

$$\begin{aligned} |p^0, p^3, m, s\rangle &= \frac{\alpha}{4\pi p_0} \{ J_m(\alpha \varrho) e^{im\phi} e^{-i(p_0 t + p_3 z)} (dz)_a \\ &\quad + \frac{i}{\sqrt{2}\alpha} [(sp_0 - p_3) J_{m-1}(\alpha \varrho) e^{im\phi} e^{-i(p_0 t + p_3 z)} (\varepsilon^-)_a \\ &\quad + (sp_0 + p_3) J_{m+1}(\alpha \varrho) e^{im\phi} e^{-i(p_0 t + p_3 z)} (\varepsilon^+)_a] \}, \end{aligned} \quad (41)$$

which satisfies

$$\langle p^0, p^3, m, s | p'^0, p'^3, m', s' \rangle = \delta(p^0 - p'^0) \delta(p^3 - p'^3) \delta_{mm'} \delta_{ss'}. \quad (42)$$

Finally, we would like to point out that $|p^0, p^3, m, s\rangle$ vanishes for $m \neq \pm 1$ in the case of $\alpha = 0$.

4.3 The Spherically Symmetrical Representation

To provide the spherically symmetrical representation, we would like to use the spherical coordinate system, i.e.

$$\begin{aligned} x^0 &= t, \\ x^1 &= r \sin \theta \cos \varphi, \\ x^2 &= r \sin \theta \sin \varphi, \\ x^3 &= r \cos \theta. \end{aligned} \quad (43)$$

In this case, the Minkowski metric takes the form

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2 \theta d\varphi^2), \quad (44)$$

and

$$\begin{aligned} P^0 &= i\left(\frac{\partial}{\partial t}\right)^a, \\ L_{\pm} \equiv L_1 \pm iL_2 &= \pm e^{\pm i\varphi}\left[\left(\frac{\partial}{\partial \theta}\right)^a \pm i \cot \theta \left(\frac{\partial}{\partial \varphi}\right)^a\right], \\ L_3 &= -i\left(\frac{\partial}{\partial \varphi}\right)^a. \end{aligned} \quad (45)$$

Define a pair of null covariant vector fields as

$$(\varepsilon^{\mp})_a = \frac{r}{\sqrt{2}}[(d\theta)_a \pm i \sin \theta (d\varphi)_a], \quad (46)$$

then from the second part in Eqn.(18), any vector potential reads

$$A_a = A_r(dr)_a + A_-(\varepsilon^-)_a + A_+(\varepsilon^+)_a, \quad (47)$$

where A_r has spin weight 0, A_- with -1 , and A_+ with 1 [17, 18, 19]. Using

$$\begin{aligned} \nabla_a(dr)_b &= \frac{1}{r}[(\varepsilon^-)_a(\varepsilon^+)_b + (\varepsilon^+)_a(\varepsilon^-)_b], \\ \nabla_a(\varepsilon^-)_b &= \frac{1}{r}\left\{\frac{\cot \theta}{\sqrt{2}}[(\varepsilon^+)_a - (\varepsilon^-)_a](\varepsilon^-)_b - (\varepsilon^-)_a(dr)_b\right\}, \\ \nabla_a(\varepsilon^+)_b &= \frac{1}{r}\left\{\frac{\cot \theta}{\sqrt{2}}[(\varepsilon^-)_a - (\varepsilon^+)_a](\varepsilon^+)_b - (\varepsilon^+)_a(dr)_b\right\}, \end{aligned} \quad (48)$$

it follows that

$$\begin{aligned}
\Box A_a &= \left[\frac{-1}{2r^2} (\bar{\partial}\bar{\partial} + \bar{\partial}\bar{\partial}) A_r + \frac{\partial^2 A_r}{\partial t^2} - \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{2}{r^2} \right) A_r - \frac{\sqrt{2}}{r^2} (\bar{\partial}A_- + \bar{\partial}A_+) \right] (dr)_a \\
&+ \left[\frac{-1}{2r^2} (\bar{\partial}\bar{\partial} + \bar{\partial}\bar{\partial}) A_- + \frac{\partial^2 A_-}{\partial t^2} - \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) A_- + \frac{\sqrt{2}}{r^2} \bar{\partial}A_r \right] (\varepsilon^-)_a \\
&+ \left[\frac{-1}{2r^2} (\bar{\partial}\bar{\partial} + \bar{\partial}\bar{\partial}) A_+ + \frac{\partial^2 A_+}{\partial t^2} - \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) A_+ + \frac{\sqrt{2}}{r^2} \bar{\partial}A_r \right] (\varepsilon^+)_a = 0,
\end{aligned} \tag{49}$$

and

$$\nabla_a A^a = -\left(\frac{\partial}{\partial r} + \frac{2}{r}\right) A_r + \frac{1}{\sqrt{2}r} (\bar{\partial}A_- + \bar{\partial}A_+) = 0. \tag{50}$$

Here $\bar{\partial}, \bar{\partial}$ are operators acting on a quantity f with spin weight n , i.e.

$$\begin{aligned}
\bar{\partial}f &= -\left(\frac{\partial}{\partial \theta} + i \csc \theta \frac{\partial}{\partial \varphi} - n \cot \theta\right) f, \\
\bar{\partial}f &= -\left(\frac{\partial}{\partial \theta} - i \csc \theta \frac{\partial}{\partial \varphi} + n \cot \theta\right) f.
\end{aligned} \tag{51}$$

Then, $\bar{\partial}f$ and $\bar{\partial}f$ have spin weight $n+1$ and $n-1$, respectively[19].

On the other hand, we have

$$\begin{aligned}
\hat{L}_\pm (dr)_a &= L_\pm^b \partial_b (dr)_a + (dr)_b \partial_a L_\pm^b = 0, \\
\hat{L}_\pm (\varepsilon^-)_a &= L_\pm^b \partial_b (\varepsilon^-)_a + (\varepsilon^-)_b \partial_a L_\pm^b = e^{\pm i\varphi} (\varepsilon^-)_a, \\
\hat{L}_\pm (\varepsilon^+)_a &= L_\pm^b \partial_b (\varepsilon^+)_a + (\varepsilon^+)_b \partial_a L_\pm^b = -e^{\pm i\varphi} (\varepsilon^+)_a,
\end{aligned} \tag{52}$$

where ∂_a is the ordinary derivative associated with the spherical coordinate system. Thus

$$\begin{aligned}
\hat{L}_\pm A_a &= (\hat{L}_\pm A_r) (dr)_a \\
&+ (\hat{L}_\pm A_- + e^{\pm i\varphi} \csc \theta A_-) (\varepsilon^-)_a \\
&+ (\hat{L}_\pm A_+ - e^{\pm i\varphi} \csc \theta A_+) (\varepsilon^+)_a.
\end{aligned} \tag{53}$$

Similarly, it is easy to check that the Lie derivatives of $\{(dr)_a, (\varepsilon^\mp)_a\}$ via $\{P^0, L_3\}$ all vanish. Whence the simultaneous eigensolutions of $\{\hat{P}^0, \hat{\mathbf{L}}^2, \hat{L}_3\}$ to Eqn.(49) with the corresponding eigenvalue $\{p^0, l(l+1), m\}$ must satisfy

$$\begin{aligned}
A_r &= R_0(r)_0 Y_{lm}(\theta, \varphi) e^{-ip_0 t}, \\
A_- &= R_-(r)_{-1} Y_{lm}(\theta, \varphi) e^{-ip_0 t}, \\
A_+ &= R_+(r)_1 Y_{lm}(\theta, \varphi) e^{-ip_0 t}.
\end{aligned} \tag{54}$$

Here ${}_nY_{lm}$ is the spin weighted spherical harmonics with l non-negative integers and $m = -l, -l + 1, \dots, l$, such that

$$\begin{aligned}
{}_0Y_{lm} &= Y_{lm}, \\
\mathfrak{D}_n Y_{lm} &= \sqrt{(l-n)(l+n+1)} {}_{n+1}Y_{lm}, \\
\bar{\mathfrak{D}}_n Y_{lm} &= -\sqrt{(l+n)(l-n+1)} {}_{n-1}Y_{lm}, \\
(\hat{L}_\pm - ne^{\pm i\varphi} \csc \theta) {}_n Y_{lm} &= \sqrt{(l \mp m)(l \pm m+1)} {}_n Y_{lm \pm 1}, \\
\hat{L}_{3n} Y_{lm} &= m_n Y_{lm},
\end{aligned} \tag{55}$$

where Y_{lm} is the ordinary spherical harmonics, and ${}_nY_{lm}$ with $l < |n|$ vanishes[19].

We next substitute Eqn.(54) into Eqn.(49) to obtain the radial equations

$$\begin{aligned}
\left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{2}{r^2} \right) R_0 + p_0^2 R_0 - \frac{l(l+1)}{r^2} R_0 + \frac{\sqrt{2l(l+1)}}{r^2} (R_- - R_+) &= 0, \\
\left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) R_- + p_0^2 R_- - \frac{l(l+1)}{r^2} R_- + \frac{\sqrt{2l(l+1)}}{r^2} R_0 &= 0, \\
\left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) R_+ + p_0^2 R_+ - \frac{l(l+1)}{r^2} R_+ - \frac{\sqrt{2l(l+1)}}{r^2} R_0 &= 0.
\end{aligned} \tag{56}$$

Furthermore, Eqn.(50) requires

$$-\left(\frac{d}{dr} + \frac{2}{r} \right) R_0 + \frac{\sqrt{l(l+1)}}{\sqrt{2r}} (R_- - R_+) = 0. \tag{57}$$

It can be shown that Eqn.(56) and Eqn.(57) are equivalent to

$$\begin{aligned}
\left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) (R_- + R_+) + p_0^2 (R_- + R_+) - \frac{l(l+1)}{r^2} (R_- + R_+) &= 0, \\
\left(\frac{d^2}{dr^2} + \frac{4}{r} \frac{d}{dr} + \frac{2}{r^2} \right) R_0 + p_0^2 R_0 - \frac{l(l+1)}{r^2} R_0 &= 0, \\
R_- - R_+ - \frac{\sqrt{2r}}{\sqrt{l(l+1)}} \left(\frac{d}{dr} + \frac{2}{r} \right) R_0 &= 0.
\end{aligned} \tag{58}$$

By the boundary condition, the solutions to Eqn.(58) are given by

$$\begin{aligned}
R_- + R_+ &= b \frac{J_{l+\frac{1}{2}}(p_0 r)}{\sqrt{p_0 r}}, \\
R_0 &= b_0 \frac{J_{l+\frac{1}{2}}(p_0 r)}{(\sqrt{p_0 r})^3}, \\
R_- - R_+ &= b_0 \frac{\sqrt{2}}{\sqrt{l(l+1)}} \left[\frac{J_{l-\frac{1}{2}}(p_0 r)}{\sqrt{p_0 r}} - l \frac{J_{l+\frac{1}{2}}(p_0 r)}{(\sqrt{p_0 r})^3} \right], \tag{59}
\end{aligned}$$

where b and b_0 are both constant coefficients; $J_{l\pm\frac{1}{2}}$ is the first kind of Bessel function of order $l \pm \frac{1}{2}$ [20].

Substituting Eqn.(59) into the eigenequations of the helicity operator with the eigenvalue $s = \pm 1$, i.e.

$$\begin{aligned}
ip_0 R_0 - s \frac{\sqrt{l(l+1)}}{\sqrt{2}r} (R_- + R_+) &= 0, \\
ip_0 R_- + s \frac{\sqrt{l(l+1)}}{\sqrt{2}r} R_0 - s \left(\frac{d}{dr} + \frac{1}{r} \right) R_- &= 0, \\
ip_0 R_+ + s \frac{\sqrt{l(l+1)}}{\sqrt{2}r} R_0 + s \left(\frac{d}{dr} + \frac{1}{r} \right) R_+ &= 0, \tag{60}
\end{aligned}$$

we obtain

$$b = isb_0 \frac{\sqrt{2}}{\sqrt{l(l+1)}}. \tag{61}$$

Note

$$\int_{4\pi} d\varphi d\theta \sin \theta_n \bar{Y}_{lmn} Y_{l'm'} = \delta_{ll'} \delta_{mm'} \tag{62}$$

with $l \geq |n|$ [19], and

$$\begin{aligned}
\int_0^\infty dr r J_{l+\frac{1}{2}}(p_0 r) J_{l+\frac{1}{2}}(p'_0 r) &= \frac{1}{p_0} \delta(p_0 - p'_0), \\
\int_0^\infty dr \frac{1}{r} J_{l+\frac{1}{2}}(p_0 r) J_{l+\frac{1}{2}}(p'_0 r) &= \frac{1}{2l+1} \left(\frac{p_0}{p'_0}\right)^{l+\frac{1}{2}}, \\
\int_0^\infty dr J_{l-\frac{1}{2}}(p_0 r) J_{l+\frac{1}{2}}(p'_0 r) &= \frac{1}{p_0} \left(\frac{p_0}{p'_0}\right)^{l+\frac{1}{2}}, \\
\int_0^\infty dr J_{l-\frac{1}{2}}(p'_0 r) J_{l+\frac{1}{2}}(p_0 r) &= 0, \\
\int_0^\infty dr J_{l-\frac{1}{2}}(p_0 r) J_{l+\frac{1}{2}}(p_0 r) &= \frac{1}{2p_0}
\end{aligned} \tag{63}$$

with $p_0 \leq p'_0$ [20]. Thus it follows that the orthonormal basis with respect to the inner product (19) in the spherical symmetrical representation reads

$$\begin{aligned}
|p^0, l, m, s\rangle &= \frac{\sqrt{l(l+1)}}{2\sqrt{r}} \left\{ \frac{J_{l+\frac{1}{2}}(p_0 r)}{p_0 r} {}_0Y_{lm}(\theta, \varphi) e^{-ip_0 t} (dr)_a \right. \\
&\quad + \frac{1}{\sqrt{2l(l+1)}} \left[\left(\frac{isp_0 r - l}{p_0 r} J_{l+\frac{1}{2}}(p_0 r) + J_{l-\frac{1}{2}}(p_0 r) \right) {}_{-1}Y_{lm}(\theta, \varphi) e^{-ip_0 t} (\varepsilon^-)_a \right. \\
&\quad \left. \left. + \left(\frac{isp_0 r + l}{p_0 r} J_{l+\frac{1}{2}}(p_0 r) - J_{l-\frac{1}{2}}(p_0 r) \right) {}_1Y_{lm}(\theta, \varphi) e^{-ip_0 t} (\varepsilon^+)_a \right] \right\},
\end{aligned} \tag{64}$$

which satisfies

$$\langle p^0, l, m, s | p'^0, l', m', s' \rangle = \delta(p^0 - p'^0) \delta_{ll'} \delta_{mm'} \delta_{ss'}. \tag{65}$$

It is obvious that $|p^0, l, m, s\rangle$ vanishes in the case of $l = 0$, which implies that the angular quantum number l of one photon only takes positive integers.

5 Discussions

We would like to stress that the framework and method presented here are also applicable to other particles with arbitrary mass and spin such as neutrino and electron. In addition, after a simple modification, our results obtained here are easy to be generalized to those cavities with the suitable boundaries, which is important not only to the investigation of the Casimir effect, but also to understanding the relationship between the holographic entropy bound and local quantum field theory[21].

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